# Dynamic Response of Systems of Mutually Synchronized Oscillators

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*Recent studies have been concerned with conditions for the stability of synchronized systems and expressions for equilibrium frequency. This paper describes the transient response of special configurations of synchronized systems of arbitrary size, as well as frequency response and jitter response for a few cases. Tentative extrapolations to more general configurations are suggested.*

## I. INTRODUCTION

Recent studies have established the sufficiency of certain rather broad conditions for the stability of linear synchronized networks,[1] and have shown that valid expressions for the equilibrium frequency of such systems can be obtained if initial conditions are taken into account.[2] Description of the transient response of such systems is of interest, but results for general configurations have not been obtained. This paper describes the results of studies of special configurations of systems of arbitrary size, and some tentative conclusions about more general configurations are suggested.

## II. SYSTEM EQUATIONS

The equations for the synchronized system will be taken in the form used by Gersho and Karafin[1] in their (9):

$$p_n'(t) = v_n(t) + h_n(t) * \sum_{m=1}^{N} a_{nm}[p_m(t - \tau_{nm}) - p_n(t)], \qquad n = 1, \cdots, N \qquad (1)$$

(where the star denotes convolution). In Laplace transformed form, assuming zero initial conditions,

$$sP_n(s) = V_n(s) + H_n(s) \sum_{m=1}^{N} a_{nm} e^{-s\tau_{nm}} P_m(s) - H_n(s)P_n(s), \qquad n = 1, \cdots, N. \tag{2}$$

In these equations, $p_n(t)$ is the phase of the oscillator at the $n$th station, $v_n(t)$ is the free-running frequency of that oscillator with the effects of local disturbances added, $h_n(t)$ is the impulse response of a control filter at the $n$th station, $\tau_{nm}$ is the delay on the transmission link from the $m$th station to the $n$th, and $a_{nm}$ is an averaging coefficient associated with that link. The coefficients are normalized so that

$$\sum_{m=1}^{N} a_{nm} = 1. \tag{3}$$

Normally, $a_{nn}$ is zero. The filter gain $H_n(s)$ has the dimensions of inverse time; its zero-frequency value, assumed to be nonnegative, is

$$H_n(0) = \lambda_n . \tag{4}$$

These equations, as pointed out by Gersho and Karafin, are conformable with the linear equations used by Karnaugh[2] if $v_n(t)$ is understood to include not only the free-running frequency of the oscillator at the $n$th station but also the sum of the transient disturbances at that station as well as some initial condition terms.

The assumption of zero initial conditions in (2) depends on the following simplifying procedure. Since only dynamic responses are to be studied, and since the equations are linear, the steady-state solution can be subtracted from the total response. Thus, $v_n(t)$ and $p_n(t)$ will be taken to represent only the disturbance component. Where the disturbance is transient, the $v_n(t)$ will be assumed to have zero values before the disturbance begins, and the initial phases will be taken as zero. The result of this procedure shows only the response to the disturbance, to which the steady-state solution would have to be added to determine the total frequencies and phases.

Although formal results for arbitrary filters and arbitrary delays will be obtained in a few cases, emphasis will be placed on the simple case of flat filters $H_n(s) \equiv \lambda_n$ (in effect, no filters) and zero transmission delays. In this case, the filter gains $\lambda_n$ determine the time scale of the response. There seems to be no compelling practical reason to make the $\lambda_n$ much greater than the reciprocal of a second. The response time can then be assumed to be large compared with the transmission delays expected in most cases as well as large enough so that the response would not be severely affected by the inherent low-pass filter

effects of ordinary electromechanical control elements. The results are sufficiently encouraging, from a practical standpoint, to suggest that it may not be necessary to incorporate filtering by design, so that the simple case appears to have some practical value.

## III. AN ELECTRICAL ANALOG; RECIPROCITY

Although explicit transient responses have been derived only for specific system configurations, it is possible to state, for systems of arbitrary configuration, a condition sufficient to guarantee that the transient response is not oscillatory. This condition is a reciprocity condition derived from the properties of a passive electrical network analog.

### 3.1 *Case 1:* $\tau_{nm} \equiv 0,\ H_n(s) \equiv \lambda_n$

Consider an electrical network as shown in Fig. 1, having $N$ nodes in addition to a ground node. A capacitor $C_n$ is connected from the $n$th node, $n = 1, \cdots, N$, to ground, and a resistor $R_{nm} = R_{mn}$ is connected between some, not necessarily all, pairs of nodes $n$, $m$. A current source delivers current from ground into each node. The Laplace-transformed node equations are

$$sE_n(s) = \frac{1}{C_n} I_n(s) + \sum_{m=1}^{N} \frac{1}{R_{nm}C_n} [E_m(s) - E_n(s)]. \tag{5}$$

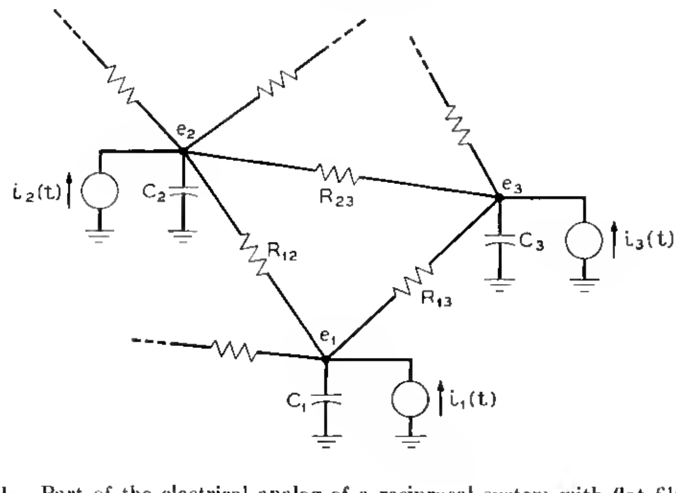


Fig. 1 — Part of the electrical analog of a reciprocal system with flat filters and zero delays.

These equations are similar in form to (2) representing a synchronized system, and can be identified with them if the $n$th node of the electrical network is identified with the $n$th station of the synchronized system, and

$$E_n(s) = P_n(s), \tag{6a}$$

$$I_n(s) = C_n V_n(s), \tag{6b}$$

$$R_{nm} = \frac{1}{a_{nm} \lambda_n C_n}, \tag{6c}$$

$$\tau_{nm} = 0, \tag{6d}$$

$$H_n(s) = \lambda_n . \tag{6e}$$

Note that $V_n(s)$ is not a voltage, but a reference frequency.

The reciprocity condition

$$R_{nm} = R_{mn} \tag{7}$$

imposes a condition on the averaging coefficients and filter gains in addition to the simplifying conditions of flat filters and zero delays. This condition immediately gives, from (6c),

$$a_{nm} \lambda_n C_n = a_{mn} \lambda_m C_m . \tag{8}$$

The capacitances $C_n$ are to a certain extent arbitrary, in that if a system has an analog with given $C_n$, equivalent analogs can be formed by multiplying all the $C_n$ by any common factor and rescaling the other elements. The capacitance at the node corresponding to any one selected station can therefore be chosen arbitrarily; (8) shows how the capacitances for stations to which it is connected can then be derived using only parameters of the synchronized system:

$$C_m = \frac{a_{nm} \lambda_n}{a_{mn} \lambda_m} C_n . \tag{9}$$

For a station that is connected to the selected one by a path of $M$ links, via $M - 1$ intermediate stations, iteration gives a formula of the form

$$C_{n_M} = \frac{a_{n_0 n_1}}{a_{n_1 n_0}} \cdot \frac{a_{n_1 n_2}}{a_{n_2 n_1}} \cdots \frac{a_{n_{M-1} n_M}}{a_{n_M n_{M-1}}} \cdot \frac{\lambda_{n_0} C_{n_0}}{\lambda_{n_M}}, \tag{10}$$

where $n$ is the index of the selected station and $n_k$ is the index of the $k$th station in sequence along the path.

Unambiguous determination of the $C_n$ requires that if two or more

paths exist between two stations, the formula (10) should give the same result on all paths. This is equivalent to the condition that the product of the averaging coefficients taken counterclockwise around any closed loop must equal the product of the averaging coefficients taken clockwise around the loop:

$$a_{n_1n_2}a_{n_2n_3} \cdots a_{n_Mn_1} = a_{n_1n_M} \cdots a_{n_3n_2}a_{n_2n_1} . \tag{11}$$

This condition will be called the reciprocity condition for synchronized systems; a system that satisfies this condition will sometimes be called a reciprocal system. It is easily seen that the reciprocity condition is both necessary and sufficient for the existence of a passive electrical analog of the form of Fig. 1, assuming that the conditions of flat filters and zero delays are also satisfied.

Since the poles of an RC network response function are all simple and lie on the negative real $s$-axis,[3] its transient response consists entirely of real exponential components. It follows immediately that a reciprocal system with flat filters and zero delays cannot have an oscillatory transient response. Moreover, errors in parameter values that cause small departures from reciprocity cannot immediately result in the appearance of oscillatory components. Such components are represented by conjugate pairs of complex poles; since the pole locations are continuous functions of the parameter values, no pole can move off the real axis until it has first moved along the axis and joined another real-axis pole to form a double pole, assuming that the departure from the reciprocal ideal is not of such form as to add new poles.

### 3.2 *Case 2: $\tau_{nm}$ small, $H_n(s)$ nearly flat*

This conclusion is strictly true only for zero delays and flat filters. However, it may be expected that delays much smaller than the system response time, or filters that are nearly flat up to frequencies much larger than the reciprocal of the response time, will have little effect on the transient response. In fact, it can be shown in specific cases that the addition of any delay, however small, introduces an infinite number of oscillatory components, which nevertheless are small in amplitude and rapidly damped so that their total effect is small. It may be assumed that the omission of delays and high frequency cutoffs is comparable to the neglect of the same parameters in ordinary circuit analysis.

It is not necessary that the filters be flat in order that the system have an electrical analog. The resistors can be replaced by any 2-terminal

networks so as to simulate any filter that has a "positive real" frequency response function. If the transfer function of the filter can be synthesized as the admittance of a network of resistors and capacitors only, the analog will still be an RC network and the transient response obviously remains nonoscillatory.

While this study is nominally confined to dynamic behavior, the Appendix shows how the reciprocity condition simplifies the steady-state analysis.

V. E. Beneš has pointed out that if the $a_{nm}$ are considered as the transition probabilities of a Markov process, as in his original study (unpublished work, 1959) of stability and equilibrium frequency, the reciprocity condition introduced here is equivalent to the condition of reversibility of the Markov process, which in turn is related to detailed balance in statistical mechanics.

## IV. TWO-STATION SYSTEMS

The analysis of a system of two stations offers not only an introduction to the techniques of analysis but also an example of the behavior of small systems for comparison with the behavior of the large systems to be described in later sections.

An impulse disturbance of frequency is assumed to occur at one of the stations, which we then designate (without loss of generality) as station 1. This form of disturbance can be interpreted as a brief rise in frequency which is almost immediately corrected, leaving a residual phase error of one unit of phase. Alternatively, it could represent any disturbance that gives rise to the sudden appearance of a phase error.

The system equations, from (2), are

$$\begin{aligned} sP_1(s) &= 1 + H_1(s)e^{-s\tau_{12}}P_2(s) - H_1(s)P_1(s), \\ sP_2(s) &= H_2(s)e^{-s\tau_{21}}P_1(s) - H_2(s)P_2(s). \end{aligned} \tag{12}$$

These equations are easily solved to give

$$\begin{aligned} P_1(s) &= \frac{s + H_2(s)}{s^2 + s[H_1(s) + H_2(s)] + H_1(s)H_2(s)[1 - e^{-s(\tau_{12}+\tau_{21})}]}, \\ P_2(s) &= \frac{H_2(s)e^{-s\tau_{21}}}{s^2 + s[H_1(s) + H_2(s)] + H_1(s)H_2(s)[1 - e^{-s(\tau_{12}+\tau_{21})}]}. \end{aligned} \tag{13}$$

The final value theorem gives

$$p_1(\infty) = p_2(\infty) = \frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_1\lambda_2(\tau_{12} + \tau_{21})} \tag{14}$$

as the ultimate displacement in phase caused by the disturbance. The equality of the two final values, signifying no net change in the phase difference between the two stations, is a necessary consequence of the uniqueness of the steady-state solution.

### 4.1 *Case 1:* $H_n(s) \equiv \lambda_n$ , $\tau_{nm} \equiv 0$.

More explicit results for the transient response are obtained in the special case of flat filters and zero delays. The transforms become simple enough for inversion by inspection; the result in the time domain is

$$
\begin{aligned}
p_1(t) &= \frac{\lambda_2}{\lambda_1 + \lambda_2} + \frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-(\lambda_1+\lambda_2)t}, \\
p_2(t) &= \frac{\lambda_2}{\lambda_1 + \lambda_2} [1 - e^{-(\lambda_1+\lambda_2)t}].
\end{aligned} \tag{15}
$$

These equations indicate a simple exponential approach to the final value, starting with initial phases [immediately after the impulse in $v_1(t)$] of 1 at the first station and 0 at the second. Such behavior appears satisfactory for a practical system.

### 4.2 *Case 2:* $H_n(s) \equiv \lambda$, $\tau_{nm} \equiv \tau$.

To determine the effect of delays, the system will be made as simple as possible in other respects. The filters will be assumed flat with equal gains, and the delays will be assumed equal. In this case, the solution (13) becomes

$$
\begin{aligned}
P_1(s) &= \frac{s + \lambda}{s^2 + 2\lambda s + \lambda^2 - \lambda^2 e^{-2s\tau}}, \\
P_2(s) &= \frac{\lambda e^{-s\tau}}{s^2 + 2\lambda s + \lambda^2 - \lambda^2 e^{-2s\tau}}.
\end{aligned} \tag{16}
$$

The denominator can be factored, and a partial expansion in partial fractions gives

$$
\begin{aligned}
P_1(s) &= \tfrac{1}{2}[Q_1(s) + Q_2(s)], \\
P_2(s) &= \tfrac{1}{2}[Q_1(s) - Q_2(s)],
\end{aligned} \tag{17}
$$

where

$$
\begin{aligned}
Q_1(s) &= \frac{1}{s + \lambda - \lambda e^{-s\tau}}, \\
Q_2(s) &= \frac{1}{s + \lambda + \lambda e^{-s\tau}}.
\end{aligned} \tag{18}
$$

One approach to the inversion of these transforms is to divide numerator and denominator by $s + \lambda$ and treat the result as the summation of a geometric series. Expansion of the series gives

$$
\begin{aligned}
Q_1(s) &= \sum_{m=0}^{\infty} \frac{\lambda^m e^{-ms\tau}}{(s + \lambda)^{m+1}} , \\
Q_2(s) &= \sum_{m=0}^{\infty} \frac{(-\lambda)^m e^{-ms\tau}}{(s + \lambda)^{m+1}} ,
\end{aligned} \tag{19}
$$

from which, by (17),

$$
\begin{aligned}
P_1(s) &= \sum_{k=0}^{\infty} \frac{\lambda^{2k} e^{-2ks\tau}}{(s + \lambda)^{2k+1}} , \\
P_2(s) &= \sum_{k=0}^{\infty} \frac{\lambda^{2k+1} e^{-(2k+1)s\tau}}{(s + \lambda)^{2k+2}} .
\end{aligned} \tag{20}
$$

Inversion term by term gives

$$
\begin{aligned}
p_1(t) &= \sum_{k=0}^{[t/2\tau]} \frac{\lambda^{2k}(t - 2k\tau)^{2k} e^{-\lambda(t-2k\tau)}}{2k!} , \\
p_2(t) &= \sum_{k=0}^{[(t-\tau)/2\tau]} \frac{\lambda^{2k+1}[t - (2k + 1)\tau]^{2k+1} e^{-\lambda[t-(2k+1)\tau]}}{(2k + 1)!} ,
\end{aligned} \tag{21}
$$

where the square bracket in the limit of summation (but only there) denotes the integer part of the enclosed expression. This result can be numerically evaluated term by term if the product $\lambda\tau$ is known. It gives an exact result (for the assumed model) up to a time depending on the number of terms evaluated. Fig. 2 shows a graph of the calculated results for $\lambda\tau = 0.1$, that is, delay equal to one-tenth of the reciprocal of the filter gain.

The interpretation of this result is that the response of each station to changes in phase at the other is delayed for a time equal to the link delay $\tau$. Thus, from $t = 0$ to $t = \tau$, station 2 is completely undisturbed. Meanwhile, from $t = 0$ to $t = 2\tau$, station 1 observes no change in the frequency received from station 2 and therefore, its response is exponential with time constant $1/\lambda$. Therefore, from $t = \tau$ to $t = 3\tau$ station 2 responds to the exponential response received from station 1, and so on. The result (21) could in fact have been derived by tracing out the response of the system in this manner.

A second approach, inherently inexact but more useful for times that are long compared to the transmission delay, is to complete the partial-fraction expansion of (18). This requires in principle determina-

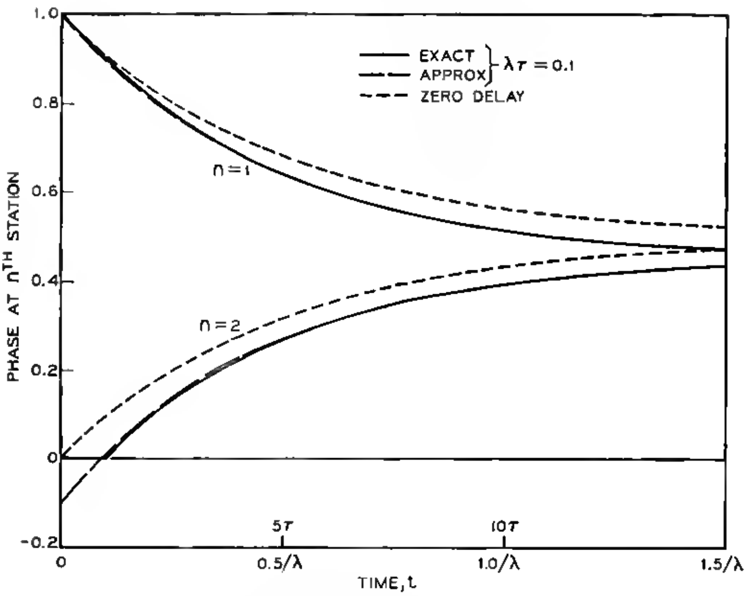


Fig. 2 — Impulse response of a two-station system.

tion of the locations of all the poles, which are determined by the transcendental equation

$$s + \lambda = \pm\lambda e^{-s\tau}, \tag{22}$$

where the "plus" sign refers to $Q_1(s)$ and the "minus" sign to $Q_2(s)$. This equation has, in general, an infinity of solutions. However, if $\lambda\tau$ is a small number, the most important components will be those due to poles of the order of magnitude of $\lambda$. The exponent in (22) is then small, so that the exponential can be approximated as

$$e^{-s\tau} \approx 1 - s\tau. \tag{23}$$

Using this approximation in (18) gives a form which is easily inverted to give, finally, from (17),

$$\begin{aligned} p_1(t) &\approx \frac{1}{2}\left[\frac{1}{1 + \lambda\tau} + \frac{1}{1 - \lambda\tau} e^{-2\lambda t/(1-\lambda\tau)}\right] \\ p_2(t) &\approx \frac{1}{2}\left[\frac{1}{1 + \lambda\tau} - \frac{1}{1 - \lambda\tau} e^{-2\lambda t/(1-\lambda\tau)}\right]. \end{aligned} \tag{24}$$

This obviously has an error at $t = 0$, which is small if $\lambda\tau$ is small, but has the correct final value determined by (14). This result is compared with the exact response, as well as with the response of a zero-delay system, in Fig. 2, which illustrates the case of $\lambda\tau = 0.1$. The approximation is better for smaller $\lambda\tau$.

## V. LARGE FULLY INTERCONNECTED SYSTEMS

Next to be considered is a network of $N$ identical stations in which all stations transmit via identical direct links to all others. All coefficients $a_{nm}$ are assumed equal:

$$a_{nm} = \frac{1}{N-1}, \qquad n = 1, \cdots, N, \qquad m \neq n. \tag{25}$$

If an impulse disturbance occurs at the first station, all the other stations display identical responses, so that

$$p_2(t) = p_3(t) = \cdots = p_N(t). \tag{26}$$

The system response can, therefore, be described in terms of two equations in $P_1(s)$ and $P_2(s)$:

$$\begin{aligned} sP_1(s) &= 1 + H(s)e^{-s\tau}P_2(s) - H(s)P_1(s) \\ sP_2(s) &= \frac{H(s)}{N-1}e^{-s\tau}[P_1(s) + (N-2)P_2(s)] - H(s)P_2(s). \end{aligned} \tag{27}$$

These equations can be formally solved to give

$$\begin{aligned} P_1(s) &= \frac{s + H(s) - \left(\frac{N-2}{N-1}\right)H(s)e^{-s\tau}}{\Delta} \\ P_2(s) &= \frac{H(s)e^{-s\tau}}{(N-1)\Delta}, \end{aligned} \tag{28}$$

where

$$\begin{aligned} \Delta = s^2 &+ sH(s)\left[2 - \left(\frac{N-2}{N-1}\right)e^{-s\tau}\right] \\ &+ H^2(s)\left[1 - \left(\frac{N-2}{N-1}\right)e^{-s\tau} - \left(\frac{1}{N-1}\right)e^{-2s\tau}\right]. \end{aligned} \tag{29}$$

The final value theorem gives

$$p_1(\infty) = p_2(\infty) = \frac{1}{N(1 + \lambda\tau)}. \tag{30}$$

5.1 *Case 1:* $H_n(s) \equiv \lambda$, $\tau_{nm} \equiv 0$.

In the special case of flat filters and no delays, inversion of the Laplace transforms gives

$$
\begin{aligned}
p_1(t) &= \frac{1}{N} + \left(1 - \frac{1}{N}\right) e^{n\lambda t/(N-1)} \\
p_2(t) &= \frac{1}{N}\left[1 - e^{-n\lambda t/(N-1)}\right].
\end{aligned}
\tag{31}
$$

In this case, the response is an exponential approach to equilibrium as in the 2-station system. When one station is disturbed, the other stations respond in unison, as one.

5.2 *Case 2:* $H_n(s) \equiv \lambda$, $\tau_{nm} \equiv \tau$

In the case of equal positive delays and flat filters, the solution (28) in transformed form can be partially expanded in partial fractions to give

$$
\begin{aligned}
P_1(s) &= \frac{1}{N}\left[Q_1(s) + (N-1)Q_2(s)\right], \\
P_2(s) &= \frac{1}{N}\left[Q_1(s) - Q_2(s)\right],
\end{aligned}
\tag{32}
$$

where

$$
\begin{aligned}
Q_1(s) &= \frac{1}{s + \lambda - \lambda e^{-s\tau}}, \\
Q_2(s) &= \frac{1}{s + \lambda + \dfrac{\lambda}{N-1} e^{-s\tau}}.
\end{aligned}
\tag{33}
$$

This is similar in form to (17) and (18), and the same methods can be used to evaluate the transient response. The principal difference between this and the 2-station case is that the conditions for cancellation of odd or even terms in the series of delayed responses (21) do not hold in the many-station case, and the antisymmetric component, $q_2(t)$, is more rapidly damped than the symmetric component $q_1(t)$. The results for the zero-delay case and for the case of $\lambda\tau = 0.1$ are shown for a 6-station system in Fig. 3.

The simplicity of both the analysis and the result can be attributed to the condition that all stations and all paths are identical. Although the effects of slight departures from this condition may be of practical

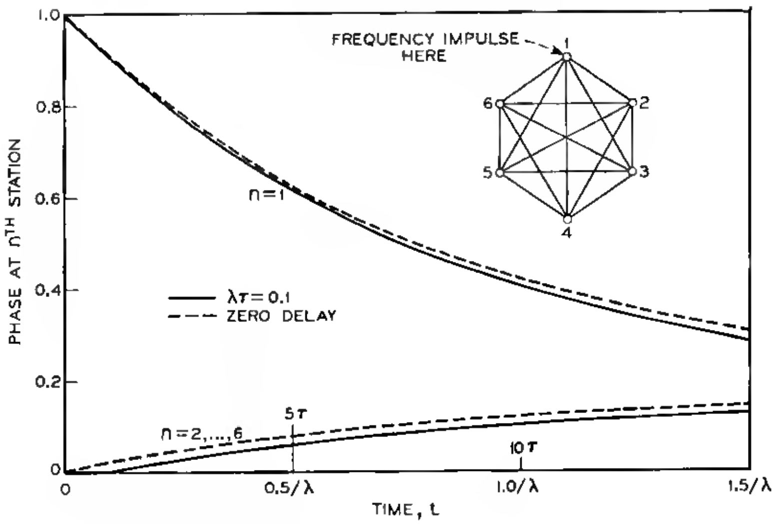


Fig. 3—Impulse response of a fully interconnected 6-station system.

interest, tbe slightest departure will destroy the symmetry and vastly complicate the analysis. As a guess, it may be supposed that the effect of a slight dissimilarity among paths will be smaller than the effect of removing some of the paths. When all but $N$ paths have been removed, in such a way that the system forms a ring in which each station receives only from its two nearest neighbors, a new form of symmetry appears, which will be used in the next section.

## VI. THE BILATERAL RING

A bilateral ring is defined as a ring of $N$ identical stations, with $2N$ identical one-way links forming $N$ two-way links by which each station sends to, and receives from, its two nearest neighbors, one on each side. This may be viewed as the opposite extreme to the fully interconnected system, providing the longest possible indirect paths in a system of $N$ identical stations. (Longer paths are possible in a chain, but the stations cannot be identical because each end station has only one neighbor.)

The equations of the bilateral ring, in transform form, are

$$sP_n(s) = V_n(s) + H(s)\{\tfrac{1}{2}[P_{n+1}(s) + P_{n-1}(s)]e^{-s\tau} - P_n(s)\},$$

$$n = 1, 2, \cdots, N, \qquad (34)$$

where addition in the index $n$ is performed modulo $N$, so that $P_{N+1}(s)$ is $P_1(s)$, $P_0(s)$ is $P_N(s)$, and the $(N - m)$th station can be alternatively designated as the $(-m)$th. Tbis system of equations will be simplified by a form of Fourier analysis. We define

$$Q_k(s) = \sum_{n=1}^{N} P_n(s) e^{-j2\pi nk/N}, \qquad k = 1, \cdots, N, \tag{35}$$

where $j$ is the imaginary unit. It can then be shown by direct substitution that

$$P_n(s) = \frac{1}{N} \sum_{k=1}^{N} Q_k(s) e^{j2\pi nk/N}, \qquad n = 1, \cdots, N. \tag{36}$$

Similarly, variables $U_k(s)$ will be defined by transformation of tbe $V_n(s)$ as in (35), with inversion as in (36). The linearity of the Laplaee transformation implies similar relations among the variables in the time domain. All these relations remain unaffected if any $n$ or $k$ is changed by adding or subtraeting $N$.

Let the $n$th equation in (34) be multiplied by $e^{-j2\pi nk/N}$, and the equation summed over all $n$. The result is

$$sQ_k(s) = U_k(s) + H(s)[\tfrac{1}{2}(e^{j2\pi k/N} + e^{-j2\pi k/N})e^{-s\tau} - 1]Q_k(s),$$
$$k = 1, \cdots, N. \tag{37}$$

This can be solved immediately to give

$$Q_k(s) = \frac{U_k(s)}{s + H(s)[1 - e^{-s\tau} \cos (2\pi k/N)]}. \tag{38}$$

Given a set of transient frequeney disturbances $v_n(t)$, one may find their Laplace transforms $V_n(s)$, find the $U_k(s)$ using (35), find the $Q_k(s)$ using (38), use (36) to obtain $P_n(s)$, and find the phase disturbances $p_n(t)$ by inverse transformation.

In the ease of an isolated impulse in frequency at the $N$th station, we have

$$V_n(s) = 0, \qquad n = 1, 2, \cdots, N - 1; \qquad V_N(s) = 1. \tag{39}$$

By using (35) we get

$$U_k(s) = 1, \qquad k = 1, \cdots, N. \tag{40}$$

Explicit solutions will be obtained here only for eases in which tbe filters are flat. Under these conditions,

$$Q_k(s) = \frac{1}{s + \lambda[1 - e^{-s\tau} \cos (2\pi k/N)]}. \tag{41}$$

The complexity of the result depends on whether the delay $\tau$ is assumed zero or positive.

6.1 *Case 1:* $H_n(s) \equiv \lambda,\ \tau_{nm} \equiv 0,\ N < \infty$.

If the delay is zero, (41) can be inverted immediately to give

$$q_k(t) = \exp\{-\lambda[1 - \cos(2\pi k/N)]t\} \tag{42}$$

and the phase disturbances, using (36), are

$$p_n(t) = \frac{1}{N}\sum_{k=1}^{N} e^{j2\pi nk/N} \exp\{-\lambda[1 - \cos(2\pi k/N)]t\} \tag{43}$$

The $N$th term in this sum is real, as is the $(N/2)$th term if $N$ is even. For all other $k$, the $k$th term is the complex conjugate of the $(N - k)$th term, so that the sum is real, and may be expressed as the sum of the the real parts of the individual terms:

$$p_n(t) = \frac{1}{N}\sum_{k=1}^{N} \cos(2\pi nk/N) \exp\{-\lambda[1 - \cos(2\pi k/N)]t\}. \tag{44}$$

The $N$th term in this sum is a constant term, which applies equally to all stations and does not affect the phase differences between stations. All other terms are real exponentials approaching zero with increasing time. The dashed curves in Figs. 4, 5, and 6 show the response of a 6-station ring calculated from (44).

6.2 *Case 2:* $H_n(s) \equiv \lambda,\ \tau_{nm} \equiv 0,\ N = \infty$.

This result can be extended to rings of indefinitely large size in two different ways, so as to specify the response either a given number of stations away from the source of the disturbance, or a given fraction of the circumference away from the source. For the first approach, which gives an exact result for an infinite ring, let

$$\theta_k = 2\pi k/N \tag{45}$$

and let $N$ increase without limit (approach infinity). Then the limit of (43) defines the integral

$$p_n(t) = \frac{e^{-\lambda t}}{2\pi}\int_0^{2\pi} e^{jn\theta} \exp(\lambda t \cos\theta)\, d\theta, \tag{46}$$

which is related to a known integral form[4] for the modified Bessel function of the first kind, order $n$, and gives

$$\begin{aligned} p_n(t) &= e^{-\lambda t} I_n(\lambda t), \qquad n = \cdots -1, 0, 1, \cdots \\ &= p_{-n}(t). \end{aligned} \tag{47}$$

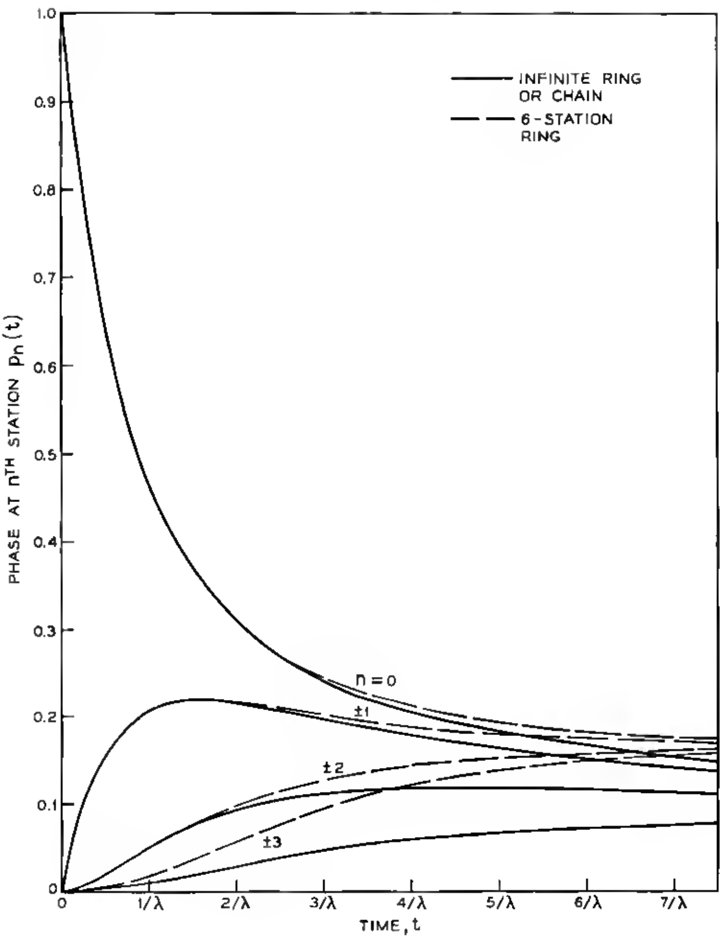


Fig. 4—Impulse response of bilateral rings with zero delays.

Curves calculated from this equation are shown as the solid curves in Fig. 4.

Full exploitation of this result requires that the station at which the disturbance originates be called the zeroth, and that neighboring stations be indexed with positive integers to one side and negative integers to the other side. At any time $t$, the largest phase disturbance is at the station at which the original disturbance occurred. The asymptotic approximation for large $x$,

$$I_n(x) \approx \frac{e^x}{\sqrt{2\pi x}} \tag{48}$$

shows that the phase disturbance decreases with increasing time roughly as

$$p_n(t) \approx \frac{1}{\sqrt{2\pi\lambda t}}. \tag{49}$$

Although this result gives the wrong limit for a finite system, it gives a clear picture of the early behavior while the response is still substantially localized.

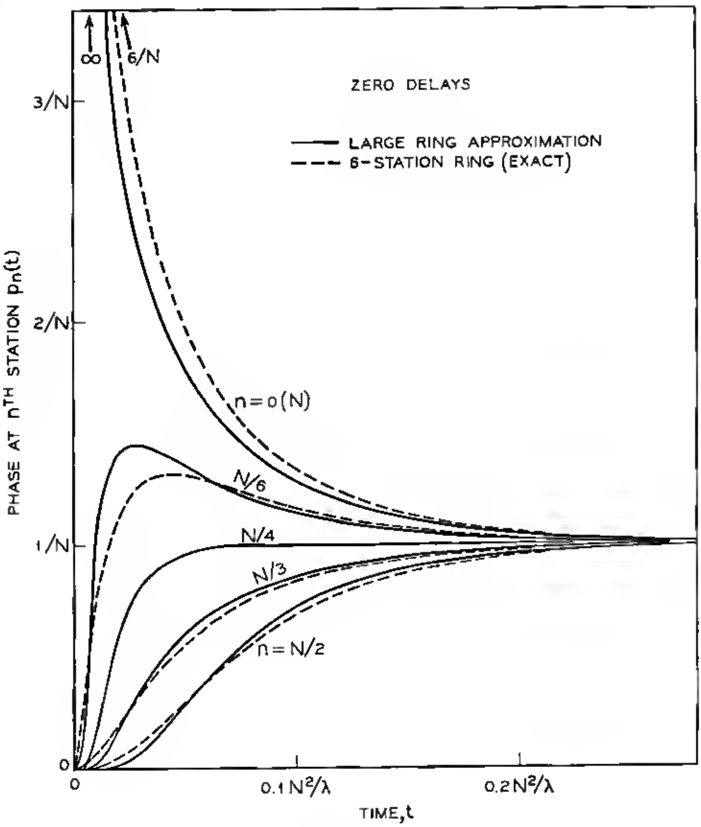


Fig. 5 — Large-$t$ approximation for a large bilateral ring.

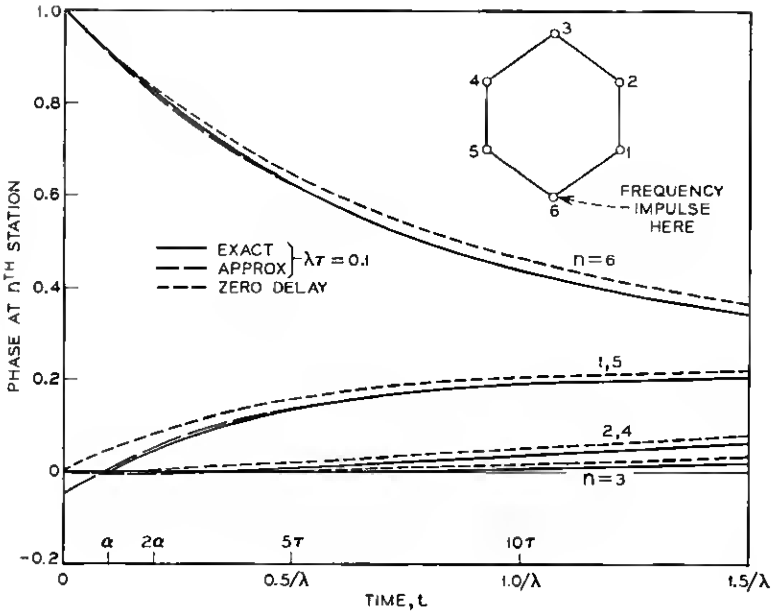


Fig. 6—Impulse response of a 6-station bilateral ring.

6.3 *Case 3:* $H_n(s) \equiv \lambda,\ \tau_{nm} \equiv 0$, *N large, t large.*

The alternative approach provides a better approximation for large $t$ after the disturbance has spread around the ring. When $t$ is large, the dominant terms in (44) are those in which $\cos(2\pi k/N)$ is closest to unity, including not only those in which $k$ is small but also those in which $k$ is close to $N$, or, equivalently, $k$ is small and negative. Since the $k$th term and the $(N - k)$th or $(-k)$th term are equal, the latter terms can be effectively included by doubling each term for small $k$. For large $N$, the approximation

$$\cos x \approx 1 - \frac{x^2}{2} \tag{50}$$

can be used for these terms. The $N$th or zeroth term is a constant $1/N$. The other terms, which are small, can be omitted or included as convenient; since it is difficult to specify in advance which terms are negligible, it seems safest to include them all, at least formally. Thus, approximately, for large $N$,

$$p_n(t) \approx \frac{1}{N} + \frac{2}{N} \sum_{k=1}^{\infty} \cos(2\pi nk/N) \exp\left(\frac{-2\pi^2 k^2}{N^2} \lambda t\right). \tag{51}$$

The time constants are proportional to the square of the number of stations in the ring. The components have sinusoidal spatial distributions around the ring, and the time constant is inversely proportional to the square of the spatial frequency. Some curves calculated from (51) are shown in Fig. 5, compared with the response of a 6-station ring.

### 6.4 *Case 4:* $H_n(s) \equiv \lambda$, $\tau_{nm} \equiv \tau$, $N < \infty$.

If the delays are positive but all equal, the methods used in the 2-station system can be applied to the inversion of (41). The exact result is

$$q_k(t) = \sum_{m=0}^{[t/\tau]} \frac{\lambda^m \cos^m (2\pi k/N)(t - m\tau)^m e^{-\lambda(t-m\tau)}}{m!}. \tag{52}$$

The approximation based on (23) gives

$$q_k(t) \approx \frac{1}{1 + \lambda\tau \cos (2\pi k/N)} \exp \left\{ -\lambda t \left[ \frac{1 - \cos (2\pi k/N)}{1 + \lambda\tau \cos (2\pi k/N)} \right] \right\} \tag{53}$$

which may be compared with (42). Curves calculated from these equations for a 6-station ring with $\lambda\tau = 0.1$ are shown in Fig. 6 and compared with the zero-delay case.

## VII. BILATERAL CHAINS

It has been mentioned previously that a chain lacks the simplicity of a ring because of the exceptional nature of the end stations. However, given any chain of $N$ stations, an analogous ring can be formed by duplicating all stations except the end stations so as to form a second chain between the end stations as shown in Fig. 7, and taking the value $\frac{1}{2}$ for each of the two averaging coefficients at each end station, leaving all other parameters unchanged. The response of the chain to a disturbance at any station can be found by applying the same disturbance at the corresponding station or stations in the analogous ring; the response of each half of the ring will be the same as the response of the original chain.

A bilateral ring, as studied in the preceding section, will result if the stations in the chain all have equal filter gains and if all averaging coefficients (except at the end stations) equal $\frac{1}{2}$. Such a chain will be called a bilateral chain. Thus, in particular, the response shown for 6-station rings in Figs. 4, 5, and 6 will also be observed in 4-station bilateral chains disturbed by an impulse at an end station. The response to a disturbance at any other station may be obtained by superposition of two station responses calculated from the ring; the responses to be

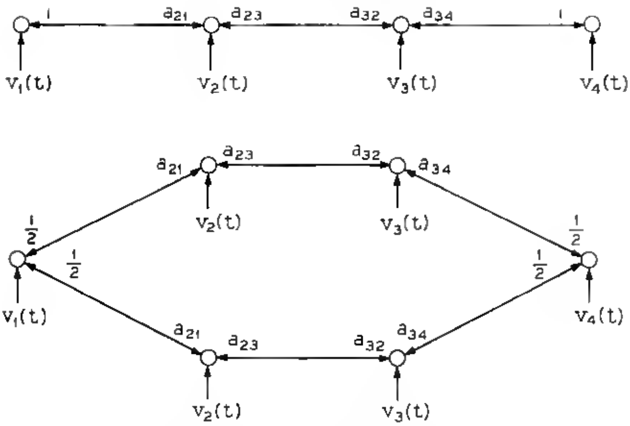


Fig. 7 — A chain and its analogous ring.

superposed may be identified by supposing that the disturbance is propagated in both directions from the source and is reflected at either end of the chain.

Alternatively, in principle, the response of either the ring or the chain might be determined by superposition of an infinite number of terms of the infinite-ring response determined in the preceding section. The response of an infinite ring is the same as that of an infinite chain extending in both directions from the source of the disturbance, since the two networks are indistinguishable. The response of a finite ring could be calculated by supposing the disturbance to propagate around the ring an unlimited number of times in both directions. For a finite chain, the disturbance could be considered to spread in both directions (except when the disturbance originates at an end station) and to be reflected whenever it reaches an end station. This method may be useful in large chains or rings as a refinement of the simple approximation of a large chain or ring as an infinite one.

## VIII. UNILATERAL RINGS AND CHAINS

All the networks studied in previous sections have satisfied the reciprocity condition, and in consequence all components of the response have been nonoscillatory: strictly so in the zero-delay case, and approximately in the case of small $\lambda\tau$. In this section, the opposite extreme is studied. In the unilateral ring, the product of the averaging coefficients in one direction is positive, while every coefficient in the other direction is zero.

### 8.1 *Rings*

To define a unilateral ring, we assume a ring of $N$ identical stations and assign a positive direction around the ring. Each station transmits only to its nearest neighbor in the positive direction. Each station then receives from only one other station and thus has only one averaging coefficient equal to unity. All links are identical. The system equations are

$$sP_n(s) = V_n(s) + H(s)[P_{n-1}(s)e^{-s\tau} - P_n(s)], \qquad n = 1, \cdots, N. \tag{54}$$

The transformation defined by (35) and (36) may be applied to this network also. In place of (41), assuming the same impulsive disturbance as given in (39), we get

$$Q_k(s) = \frac{1}{s + \lambda(1 - e^{-s\tau - j2\pi k/N})}. \tag{55}$$

#### 8.1.1 *Case 1:* $H_n(s) \equiv \lambda,\ \tau_{nm} \equiv 0,\ N < \infty$

Only the case of zero delays has been studied in detail. In this case,

$$q_k(t) = \exp\{-\lambda[1 - \cos(2\pi k/N)]t\} \exp[-j\lambda t \sin(2\pi k/N)]. \tag{56}$$

Hence,

$$p_n(t) = \frac{1}{N} \sum_{k=1}^{N} \exp\{j[2\pi nk/N - \lambda t \sin(2\pi k/N)]\} \cdot \exp\{-\lambda[1 - \cos(2\pi k/N)]t\}. \tag{57}$$

The sum is real and may be alternatively expressed as

$$p_n(t) = \frac{1}{N} \sum_{k=1}^{N} \cos[2\pi nk/N - \lambda t \sin(2\pi k/N)] \cdot \exp\{-\lambda[1 - \cos(2\pi k/N)]t\}. \tag{58}$$

The components are not real exponentials, but exponentially damped sinusoids.

#### 8.1.2 *Case 2:* $H_n(s) \equiv \lambda,\ \tau_{nm} \equiv 0,\ N = \infty$

In the infinite unilateral ring, using (45) in (57) and passing to the limit,

$$p_n(t) = \frac{e^{-\lambda t}}{2\pi} \int_0^{2\pi} e^{in\theta} \exp(\lambda t e^{-i\theta})\, d\theta. \tag{59}$$

Expanding $\exp(\lambda t e^{-i\theta})$ as a power series in $\lambda t e^{-i\theta}$ and integrating term by term gives

$$\begin{aligned} p_n(t) &= \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \qquad n = 0, 1, \cdots \\ &= 0, \qquad\qquad\quad n = -1, -2, \cdots . \end{aligned} \tag{60}$$

This result is plotted in Fig. 8. The phase disturbances at adjacent stations are in the ratio

$$\frac{p_n(t)}{p_{n-1}(t)} = \frac{\lambda t}{n}, \tag{61}$$

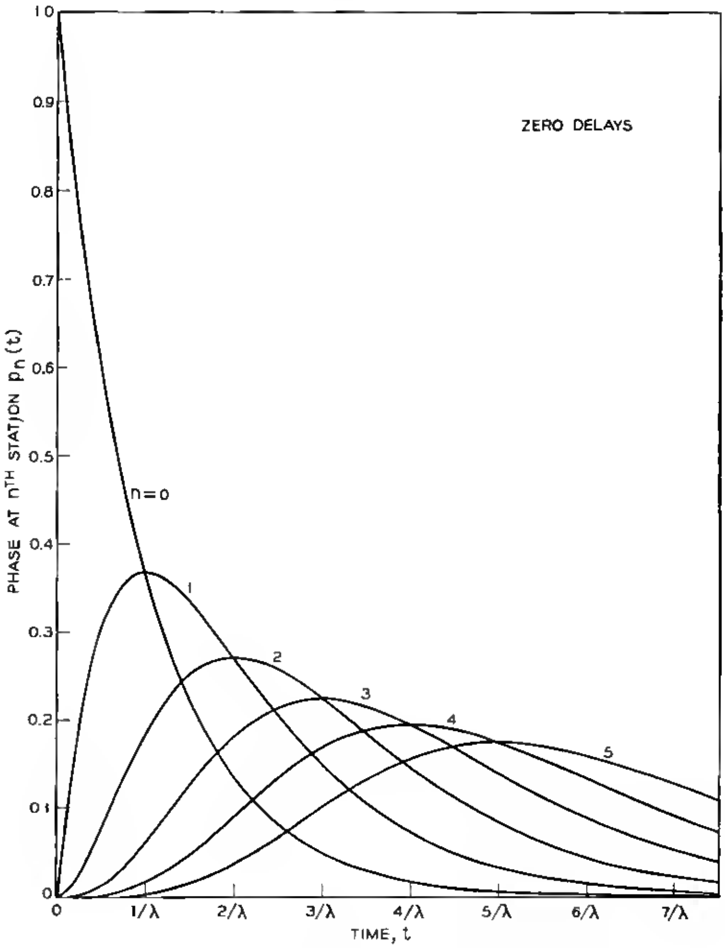


Fig. 8 — Impulse response of an infinite unilateral ring.

so that for fixed $t$, and increasing $n$, $p_n(t)$ increases until $n \geqq t$. Therefore, at any time $t$ the largest disturbance is at the $m$th station, where

$$\lambda t - 1 \leqq m \leqq \lambda t. \tag{62}$$

For large $t$, the magnitude of the largest phase disturbance, obtained with the aid of Stirling's approximation for the factorial, is asymptotically

$$p_m(t) \sim \frac{1}{\sqrt{2\pi\lambda t}}. \tag{63}$$

This is the same as the asymptotic value (49) obtained for the infinite bilateral ring, except that in the unilateral case the peak precesses at the rate of $\lambda$ stations per unit time.

8.1.3 *Case 3: $H_n(s) \equiv \lambda$, $\tau_{nm} \equiv 0$, $N$ large, $t$ large*

Application of the approximation (50) together with

$$\sin x \approx x \tag{64}$$

gives, for large $t$ in a large ring,

$$p_n(t) \approx \frac{1}{N} + \frac{2}{N} \sum_{k=1}^{\infty} \cos\left[2\pi k(n - \lambda t)/N\right] \exp\left(\frac{-2\pi^2 k^2}{N^2} \lambda t\right). \tag{65}$$

Compared with (51) this shows a response that resembles tbat of a bilateral ring except tbat it precesses around tbe ring in the positive direction at the rate of $\lambda$ stations per unit time. The oscillatory nature of the response is associated with the progression of the disturbance around the ring.

8.2 *Chains*

A finite unilateral chain is a system with one master station and $N - 1$ slaves. If such a system is disturbed at one of the slave stations, the response of each station following it in the positive direction is the same as that of tbe corresponding station in an infinite unilateral ring. An impulse disturbance at the master station, however, does not correspond directly to any situation in a unilateral ring. A permanent phase shift of one unit occurs in the master station output. Tbe effect at the second station is the same as that of a step of magnitude $\lambda$ in the free-running frequency of the second station, and, a step being the integral of an impulse, the response of tbe entire chain can be inferred by integrating the response to an impulse at the second station. Each station can thus be shown to approach its new equilibrium phase

monotonically. If the nominal time of response of each station is defined as the time of maximum rate of change of phase (maximum frequency shift), each station responds with a delay of $1/\lambda$ after the preceding station. The effect of positive link delays in the unilateral chain is to further delay the response without changing its form.

The unilateral ring is not the analog of any chain in the sense of the preceding section.

## IX. RECTANGULAR ARRAYS

A rectangular array, in which each station is connected to four nearest neighbors, can be considered as intermediate between a fully interconnected system and a chain or ring, and may be more appropriate than either as a model of a network of stations on the surface of the earth. A rectangular network with no edges or corners can be laid out on the surface of a toroid as in Fig. 9. This network can be analyzed by methods similar to those used for rings.

The stations are most conveniently indexed with double subscripts, $m = 1, \cdots, M_1$, and $n = 1, \cdots, M_2$; the number of stations is $N = M_1M_2$. Assuming equal filters and equal delays, the system equations are

$$\begin{aligned} sP_{mn}(s) &= V_{mn}(s) - H(s)P_{mn}(s) \\ &+ \frac{H(s)}{4} e^{-s\tau}[P_{m,n-1}(s) + P_{m,n+1}(s) + P_{m-1,n}(s) + P_{m+1,n}(s)], \end{aligned} \tag{66}$$

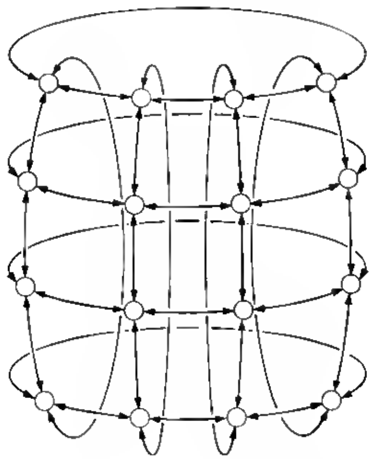

Fig. 9 — A toroidally-connected rectangular array.

assuming addition modulo $M_1$ in the first index, and modulo $M_2$ in the second. Defining

$$Q_{kl}(s) = \sum_{m=1}^{M_1} \sum_{n=1}^{M_2} P_{mn}(s) \exp\left[-j2\pi\left(\frac{mk}{M_1} + \frac{nl}{M_2}\right)\right],$$

$$k = 1, \cdots, M_1, \qquad l = 1, \cdots, M_2 \tag{67}$$

and proceeding as with the bilateral ring, we obtain, in the case of flat filters and zero delays, with an impulse disturbance at the $M_1$, $M_2$th (or zero-zeroth) station,

$$p_{mn}(t) = \frac{1}{N} \sum_{k=1}^{M_1} \sum_{l=1}^{M_2} \exp\left[j2\pi\left(\frac{mk}{M_1} + \frac{nl}{M_2}\right)\right]$$

$$\cdot \exp\left[-\lambda\left(1 - \tfrac{1}{2}\cos\frac{2\pi k}{M_1} - \tfrac{1}{2}\cos\frac{2\pi l}{M_2}\right)t\right]. \tag{68}$$

Comparison with the bilateral ring is most convenient in the limiting cases of large systems. For the infinite array,

$$p_{mn}(t) = e^{-\lambda\tau} I_m\left(\frac{\lambda t}{2}\right) I_n\left(\frac{\lambda t}{2}\right) \tag{69}$$

which has the asymptotic form

$$p_{mn}(t) \sim \frac{1}{\pi\lambda t} \tag{70}$$

indicating a more rapid approach to the final value in the rectangular array than in the ring. The approximation (50) for large $t$ in large arrays gives

$$p_{mn}(t) \approx \frac{1}{N} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \cos\left[2\pi\left(\frac{mk}{M_1} + \frac{nl}{M_2}\right)\right]$$

$$\cdot \exp\left[-\pi^2\lambda t\left(\frac{k^2}{M_1^2} + \frac{l^2}{M_2^2}\right)\right]. \tag{71}$$

The longest time constant is shorter for a rectangular array than for a ring with the same number of stations. Fig. 10 shows some curves calculated from (69).

A bounded rectangular array in a plane is more complicated than a toroidally connected array because of the exceptional edge and corner stations. However, a bounded $M_1$ by $M_2$ array can be analyzed in terms of an analogous $2M_1 - 2$ by $2M_2 - 2$ toroidal array as shown in Fig. 11. All columns except the first and last are duplicated and connected as shown by the solid lines to form a cylindrical array, and

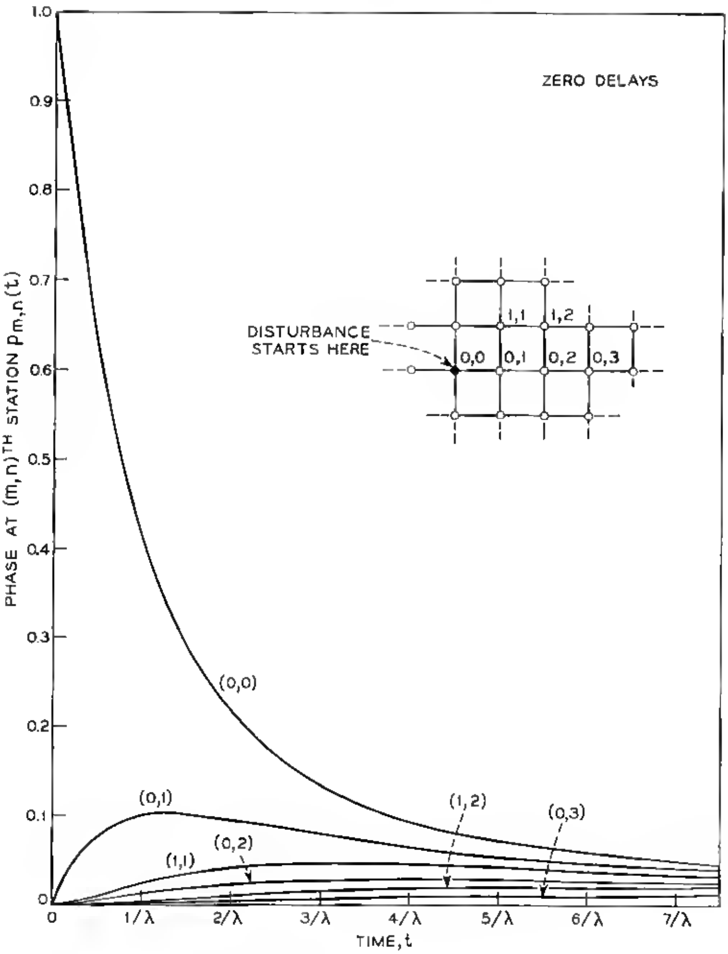


Fig. 10 — Impulse response of an infinite rectangular array.

then all rows except the first and last are duplicated and connected as shown by the dashed lines; averaging coefficients are divided by two whenever a station receives from duplicate stations. The toroidal array has one station corresponding to each original corner station, two for each edge station, and four for each interior station. The response of the original bounded array to a disturbance at any station is identical with the response of the corresponding part of the toroid when the original disturbance is applied to corresponding stations.

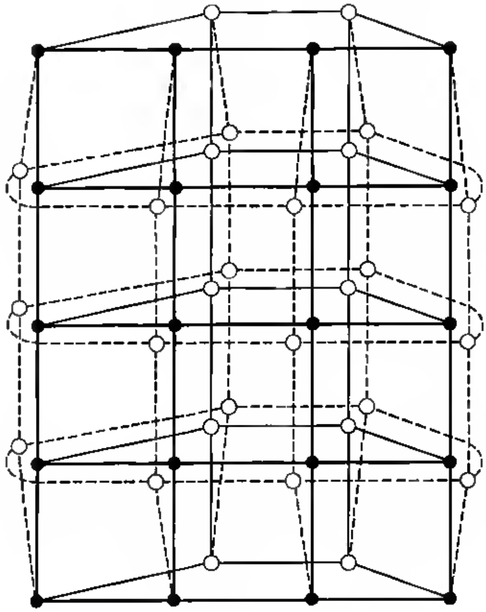

Fig. 11—The toroidally-connected array analogous to a bounded rectangular array.

Alternatively, in principle, the response of a finite toroidal array can be determined from that of the infinite array by considering the response to propagate around a toroidal array in the four cardinal directions, or to be reflected from the sides of a bounded array.

## X. RESPONSE TO SINUSOIDAL DISTURBANCES

The steady-state response of a linear system to a sinusoidal disturbance is sinusoidal, and the phase difference between the response and the input disturbance, together with the ratio of the amplitudes, is given by the frequency response function as a function of frequency. The impulse response is equivalent in principle to the frequency response function as a specification of dynamic properties, since either can be expressed in terms of the other through Fourier or Laplace transformation. The frequency response functions of a bilateral ring, in particular, are the functions $P_n(j\omega)$, which are the $P_n(s)$ evaluated along the "real frequency axis" $s = j\omega$, for real $\omega$.

The frequency response will be determined in this section for infinite rings, both bilateral and unilateral, in the case of arbitrary equal

filters and arbitrary equal delays. Although the expressions are more complicated than the impulse response expression in the case of flat filters and zero delays, they do not become very much more complicated in the more general case, for which closed form expressions for the impulse response have not been obtained.

10.1 *Case 1: Bilateral Ring, $N = \infty$*

For the bilateral ring, an expression for $Q_k(s)$ is obtained from (38) using (40), and $P_n(s)$ is obtained using (36). Using the substitution (45) and passing to the limit of infinite $N$ gives

$$P_n(s) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{jn\theta}\, d\theta}{s + H(s)(1 - e^{s\tau} \cos \theta)}. \tag{72}$$

To evaluate this by contour integration, let

$$z = e^{j\theta}. \tag{73}$$

Then

$$P_n(s) = \frac{1}{j2\pi} \oint \frac{z^{n-1}\, dz}{s + H(s)\left[1 - \frac{e^{-s\tau}}{2}\left(z + \frac{1}{z}\right)\right]}, \tag{74}$$

integrated counterclockwise around the unit circle centered at the origin in the $z$-plane.

When $n \geqq 0$ the integrand has two poles in the $z$-plane, located at the roots of the quadratic equation

$$z^2 - 2e^{s\tau}\left[1 + \frac{s}{H(s)}\right]z + 1 = 0. \tag{75}$$

Since the denominator of the integrand is symmetric in $z$ and $1/z$, one root is the reciprocal of the other. We defer consideration of the case where both roots have unit magnitude; then one pole will lie inside the path of integration and the other outside. Denote the root inside the contour by

$$z_1 = e^{s\tau}\left[1 + \frac{s}{H(s)}\right] - \sqrt{e^{2s\tau}\left[1 + \frac{s}{H(s)}\right]^2 - 1}, \tag{76}$$

where it is understood that the square root is to be taken to have whichever sign gives $z_1$ the smaller magnitude.

For convenience, let

$$\beta(s) = \frac{H(s)}{s + H(s)}, \tag{77}$$

this incidentally being the quantity wbose magnitude is required to be less than unity in the sufficient condition for stability given by Gersho and Karafin.[1] Then

$$z_1 = \frac{1 - \sqrt{1 - \beta^2(s)e^{-2s\tau}}}{\beta(s)e^{-s\tau}} \tag{78}$$
$$= \frac{\beta(s)e^{-s\tau}}{1 + \sqrt{1 - \beta^2(s)e^{-2s\tau}}},$$

where the second form can be obtained by rationalizing the numerator. The integral around the contour is $2\pi j$ times the residue at this pole, so that $P_n(s)$ is equal to the residue, which can be put in the alternative forms

$$P_n(s) = \frac{[1 - \sqrt{1 - \beta^2(s)e^{-2s\tau}}\,]^n}{H(s)\beta^{n-1}(s)e^{-ns\tau}\sqrt{1 - \beta^2(s)e^{-2s\tau}}} \tag{79}$$
$$= \frac{\beta^{n+1}(s)e^{-ns\tau}}{H(s)\sqrt{1 - \beta^2(s)e^{-2s\tau}}[1 + \sqrt{1 - \beta^2(s)e^{-2s\tau}}\,]^n}.$$

For negative $n$, (74) can be transformed into an integral in $y = 1/z$ to show that

$$P_n(s) = P_{-n}(s). \tag{80}$$

The deferred case in which $z_1$ has unit magnitude will now be briefly considered. In this case, the quadratic equation (75) has two conjugate roots of unit magnitude, or double roots at 1 or $-1$, and it is easily shown that this occurs when $\beta(s)e^{-s\tau}$ is real and has magnitude 1 or greater. If the sufficient condition for stability mentioned earlier is satisfied, this cannot occur in the left half $s$-plane or on the real frequency axis except at zero frequency, where a singularity is expected to occur in any system configuration.

Where $\beta(s)e^{-s\tau}$ is $\pm 1$, $P_n(s)$ is infinite and will ordinarily have a branch point. This always occurs at $s = 0$, and occurs for other values depending on the filters and delays. Where $\beta(s)e^{-s\tau}$ is real and has magnitude greater than unity, $P_n(s)$ will be finite but will have a step discontinuity, because as $s$ passes through a value at which $\beta(s)e^{-s\tau}$ is real, $z_1$ crosses the unit circle and must immediately be redefined as $z_2$, and the square roots in (79) abruptly change sign. The function $P_n(s)$ is thus defined as a single-valued function in the $s$-plane with line discontinuities where it might be expected to have branch cuts.

If the system is stable, these discontinuities are confined to the interior of the left half $s$-plane except for $s = 0$.

Thus, (79) defines $P_n(j\omega)$ as a continuous single-valued function except at $\omega = 0$. In the case of flat filters and zero delays, $P_n(s)$ is the Laplace transform of (47). Fig. 12 shows the magnitude of $P_n(j\omega)$ for this case and for the case of $\lambda\tau = 0.1$.

### 10.2 *Case 2: Unilateral Ring, $N = \infty$*

For an infinite unilateral ring, a similar procedure gives

$$P_n(s) = \frac{1}{j2\pi} \oint \frac{z^n \, dz}{[s + H(s)]z - H(s)e^{-s\tau}} \tag{81}$$

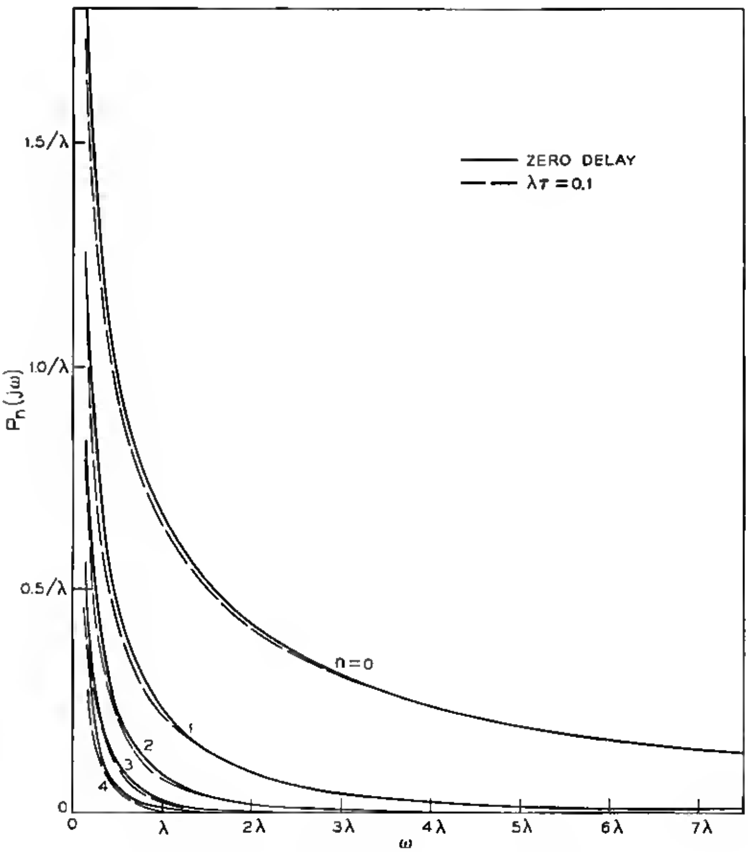


Fig. 12 — Frequency response of an infinite bilateral ring.

to be integrated over the same path as (74). When $n \geqq 0$, the integrand has a single pole at $z = \beta(s)e^{-s\tau}$. If the magnitude of $\beta(s)e^{-s\tau}$ is less than 1, the pole is inside the unit circle, and for nonnegative $n$

$$P_n(s) = \frac{\beta^{n+1}(s)e^{-ns\tau}}{H(s)}, \qquad n = 0, 1, 2, \cdots, \tag{82}$$

while for negative $n$ under the same conditions, the substitution $y = 1/z$ puts all poles outside the unit circle and

$$P_n(s) = 0, \qquad n = -1, -2, \cdots. \tag{83}$$

Fig. 13 shows the magnitude of $P_n(j\omega)$ graphically. As the magnitude of $\beta(s)e^{-s\tau}$ becomes greater than 1, the pole crosses the unit circle and there is a step discontinuity in $P_n(s)$ for all $n$. However, the sufficient condition for stability mentioned previously is both necessary and sufficient, in the unilateral ring, for these discontinuities to be confined to the left half-plane.

The finite value of $P_n(0)$, where a singularity should occur, is attributable to the fact that every station in the infinite unilateral ring is a slave station, and no finite change at any given station can alter the equilibrium frequency. The infinite unilateral ring is in this sense a pathological limiting case of the unilateral chain in which the master station recedes to infinity and becomes inaccessible.

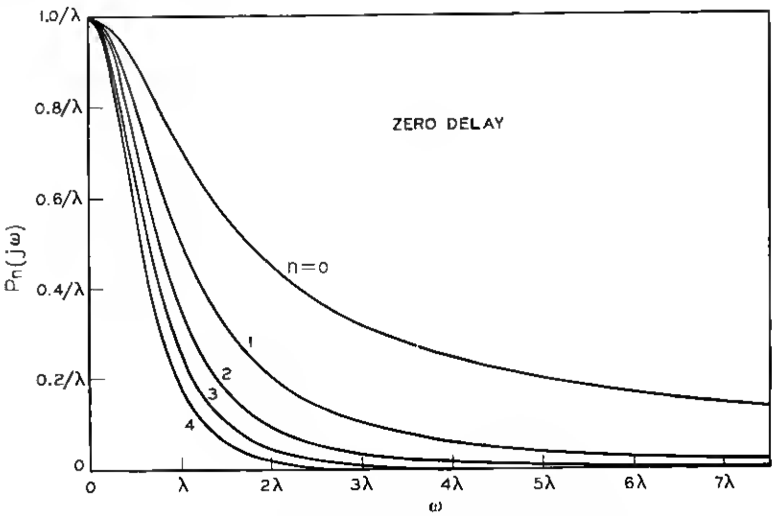


Fig. 13 — Frequency response of an infinite unilateral ring.

## XI. JITTER RESPONSE

Jitter denotes random variations in the phase of a signal. In a digital signal, jitter can occur as a result of pattern-induced retiming errors in regenerative repeaters.[5] Jitter reducers[6] can reduce the high-frequency components of jitter, but, because the jitter reducer output frequency is slaved to the input frequency, the low-frequency components cannot be reduced.

In a mutually synchronized system, the low-frequency components of jitter will affect the observed phase differences used to control the clocks. Even if the variations in the received phase, after jitter reduction, are not themselves objectionable, they might cause objectionable variations in clock phases through the cumulative effects of each clock acting on the next. To simplify the analysis, only the effects on the clock phases are considered; the immediate effects of jitter are omitted.

It is assumed that the effect of jitter on the link from the $m$th station to the $n$th is to add a random component $\mu_{nm}(t)$ to the phase $p_m(t - \tau_{nm})$ that would be received without jitter. This random component is assumed to have the properties of white Gaussian noise and to be independent on different links. Assuming that a jitter reducer can be designed that will compensate the immediate effects of jitter, we determine only the cumulative effect of jitter propagating through the system as a result of its effect on the station clocks.

The autocorrelation function assumed for $\mu_{nm}(t)$ is

$$E[\mu_{nm}^*(t)\mu_{nm}(t + x)] = K\delta(x). \tag{84}$$

Here "$E$" stands for the "expectation" or mean value, the star denotes complex conjugation (immaterial here since $\mu_{nm}$ is real), and $\delta(t)$ is the Dirac delta function. $K$ represents the noise power density, assumed to be the same for every link in the systems to be considered.

### 11.1 *Case 1: Phase-Locked Oscillator*

As a standard of comparison, consider the effect of this jitter on a simple phase-locked loop of gain $\lambda$, in which an oscillator is controlled by the signal received from an unperturbed source over a jittered link. The equation for the output phase $p(t)$ in this system is

$$p'(t) = F_1 + \lambda(F_0 t + \mu(t) - p(t)), \tag{85}$$

where $F_1$ is the free-running frequency of the controlled oscillator, $F_0$ the frequency of the master source, and the link delay is assumed zero. Since the system is linear, and we are interested only in the random

component of the output, we may set $F_1 = F_0 = 0$ without loss of generality. Thus, the Laplace-transformed system equation becomes

$$sP(s) = \lambda M(s) - \lambda P(s) \tag{86}$$

with solution

$$P(s) = \frac{\lambda}{s + \lambda} M(s). \tag{87}$$

We obtain the mean-square value of $p(t)$ from its autocorrelation function $\varphi(x)$ evaluated at $x = 0$. This is determined from the power-density spectrum

$$\Phi(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(x) e^{-i\omega x} \, dx \tag{88}$$

by means of the inverse transformation

$$\varphi(x) = \int_{-\infty}^{\infty} \Phi(\omega) e^{i\omega x} \, d\omega. \tag{89}$$

The power-density spectrum of the input $\mu(t)$, obtained from (84) and an integral of the form of (88), is flat, equal to $K/2\pi$, for all $\omega$. The output power density is obtained by multiplying this by the square of the magnitude of the frequency response, obtained from (87):

$$\Phi(\omega) = \frac{\lambda^2 K}{2\pi \mid j\omega + \lambda \mid^2}. \tag{90}$$

The inversion integral (89) is evaluated by means of a partial-fraction expansion. The analytic continuation of (90) in the $s$-plane, $s = j\omega$, has poles in both the right and left half-planes. Since (89) is a Fourier (not Laplace) inversion, terms due to poles in the right half-plane will be zero for positive $x$; thus, for positive $x$ we need only consider the left half-plane. We obtain

$$\varphi(x) = \frac{\lambda K e^{-\lambda x}}{2} \tag{91}$$

and, as a limit,

$$\varphi(0) = \frac{\lambda K}{2} \tag{92}$$

is the mean-square value of $p(t)$. The rms phase error is of course the square root of this.

### 11.2 *Case 2: Bilateral Ring*

We now consider a bilateral ring with flat filters and small delays. Each station receives two inputs, each with jitter with the autocor-

relation function (84), and therefore power density $K/2\pi$. Each input is multiplied by $\lambda/2$, to produce an error-signal component with power density $(\lambda/2)^2K/2\pi$, and when two independent components are added, their power densities add to produce a total power density of $2(\lambda/2)^2K/2\pi$, or $\lambda^2K/4\pi$. The effect of the two jitter components is thus equal to the effect of a white noise component of power density $\lambda^2K/4\pi$ added to the free-running frequency. We ignore the steady-state components and consider this to be the only input at each station. Thus, we assume

$$E[v_n^*(t)v_n(t+x)] = \frac{\lambda^2K}{2}\,\delta(x). \tag{93}$$

The variables $u_k(t)$ are derived from the $v_n(t)$ as in (84); direct evaluation gives

$$E[u_k^*(t)u_l(t+x)] = \begin{cases} \dfrac{\lambda^2K}{2N}\,\delta(x), & k = l; \\ 0, & k \neq l. \end{cases} \tag{94}$$

This shows that the $u_k(t)$ are uncorrelated, and therefore (since we have assumed Gaussian distributions) independent, each with power density $\lambda^2K/4\pi N$. It follows, since each $q_k(t)$ depends only on the corresponding $u_k(t)$ as in (38), that the $q_k(t)$ are independent. Denoting their autocorrelation functions by $\psi_k(x)$, we obtain their power-density spectra, using the frequency response given by (38), as

$$\Psi_k(\omega) = \frac{\lambda^2K/4\pi N}{[\lambda + j\omega - \lambda e^{-j\omega\tau}\cos\theta_k][\lambda - j\omega - \lambda e^{j\omega\tau}\cos\theta_k]}, \tag{95}$$

where the substitution (45) is used as an abbreviation.

When $k = N$, (95) indicates infinite power density at zero frequency. The autocorrelation function $\psi_N(x)$ is consequently infinite for all $x$, and in particular the mean-square value of $q_N(t)$ is infinite, so that the mean-square value of each $p_n(t)$ is infinite. This occurs because the random variations that the jitter induces in the system frequency cause the system phase to execute a random walk. However, since $q_N(t)$ contributes equally to every $p_n(t)$, it does not affect the phase differences between clocks, and all other $q_k(t)$ have finite mean-square values. It follows that while the phase of each clock tends to deviate indefinitely far from that of an unperturbed clock of the same frequency, the deviation between clocks in the same system tends to remain bounded.

We are primarily interested in the phase difference between the clock at each station and the delayed signal received from an adjacent station. The mean-square value of this phase difference will be denoted by

$$\Phi_{n,n\pm 1} = E\{[p_n(t) - p_{n\pm 1}(t - \tau)]^2\}. \tag{96}$$

We express the phases in terms of $q_k(t)$ using (36), writing the square of the real sum of these complex quantities as the product of the sum by its conjugate so that expansion of the product gives terms of the form of the left side of (84); thus,

$$\Phi_{n,n\pm 1} = 2 \sum_{k=1}^{N-1} \{\psi_k(0) - \mathrm{Re}\,[e^{\mp j\theta_k}\psi_k(\tau)]\}. \tag{97}$$

Therefore, (95) should be used in an integral of the form of (89) to determine $\psi_k(0)$ and $\psi_k(\tau)$. The analytic continuation of (95) in the $s$-plane has, in the left half-plane, all the poles of (41), and in addition the reflections of these poles in the right half-plane. We continue to use (23) as an approximation when $\lambda\tau$ is small. The result is

$$\psi_k(x) \approx \frac{\lambda K \exp\left[-\lambda x\left(\dfrac{1 - \cos\theta_k}{1 + \lambda\tau\cos\theta_k}\right)\right]}{4N(1 - \cos\theta_k)(1 + \lambda\tau\cos\theta_k)}, \qquad x \geqq 0. \tag{98}$$

In particular, to the first order in $\lambda\tau$,

$$\psi_k(0) \approx \frac{\lambda K(1 - \lambda\tau\cos\theta_k)}{4N(1 - \cos\theta_k)} \tag{99}$$

and, again using the linear approximation to the exponential,

$$\psi_k(\tau) \approx \frac{\lambda K(1 - \lambda\tau)}{4N(1 - \cos\theta_k)}. \tag{100}$$

We now find, from (97), that

$$\Phi_{n,n\pm 1} \approx \left(\frac{N-1}{N}\right)\frac{\lambda K}{2}. \tag{101}$$

The mean-square phase discrepancy observed in received signals is thus substantially independent of the size of the system and substantially unaffected by small link delays. It is roughly equal to the mean-square phase error, given by (92), that would be induced, by the jitter in a single link, in a simple phase-locked oscillator with control gain $\lambda$.

### 11.3 *Case 3: Unilateral Ring*

In a unilateral ring, each station receives only one input, so that the equivalent $v_n(t)$ has power density $\lambda^2 K/2\pi$ and its autocorrelation function has twice the value given in (93). The appropriate frequency response is given by (55), so that instead of (95) we get

$$\Psi_k(\omega) = \frac{\lambda^2 K/2\pi N}{[\lambda + j\omega - \lambda e^{-j\omega\tau - j\theta_k}][\lambda - j\omega - \lambda e^{j\omega\tau + j\theta_k}]}. \tag{102}$$

We continue to use (23) to determine a simple approximation. The result is

$$\psi_k(x) \approx \frac{\lambda K \exp\left[-\lambda x\left(\frac{1 - e^{-j\theta_k}}{1 + \lambda\tau e^{-j\theta_k}}\right)\right]}{2N(1 - \lambda\tau)(1 - \cos\theta_k)}. \tag{103}$$

In particular,

$$\psi_k(0) \approx \frac{\lambda K(1 + \lambda\tau)}{2N(1 - \cos\theta_k)} \tag{104}$$

and

$$\psi_k(\tau) \approx \frac{\lambda K[1 + \lambda\tau e^{-j\theta_k}]}{2N(1 - \cos\theta_k)}. \tag{105}$$

Equation (97) is equally valid for the unilateral ring as for the bilateral ring, giving

$$\begin{aligned} \Phi_{n,n-1} &\approx \lambda K\left(\frac{N-1}{N}\right) \\ \Phi_{n,n+1} &\approx \lambda K\left[\left(\frac{N-1}{N}\right) + 2\lambda\tau\left(\frac{N-2}{N}\right)\right]. \end{aligned} \tag{106}$$

The mean-square phase discrepancy is essentially twice that which occurs in the bilateral ring. The link delay has a first-order effect on the signal received at each station from the station to which it transmits timing control because of the round-trip delay.

## XII. SUMMARY AND CONCLUSIONS

In this section, I propose to extrapolate the specific results of the preceding sections to general conclusions that, although not strictly proven, seem quite likely to be true from a practical standpoint.

It was shown in Section III that a system that satisfies the reciprocity condition and has flat filters and no delays will have a nonoscillatory transient response. The response was described more specifically in later sections for specific configurations: 2-station systems, fully interconnected systems, and bilateral rings and chains, all of which met these conditions. These configurations appear to span the extremes of practical systems.

The effect of delays was determined specifically only for these special configurations and for the special case of equal delays and flat filters. The effect was shown graphically for $\lambda\tau = 0.1$; it appeared to be small and unobjectionable. Fig. 6 shows that at all stations, over the time range shown, the response to a transient disturbance is actually smaller when delays exist. At the zeroth (or $N$th) station, where the disturbance originates, this can be attributed to the short period after the disturbance during which the neighboring stations remain undisturbed and are, therefore, reliable indicators of the original state. At other stations, the appearance of smaller disturbances is due in part to the delayed peak of the response.

I propose to conjecture that the dynamic effect of delay in any reciprocal system with flat filters will be equally unobjectionable as long as the product of the largest filter gain and the largest single link delay is less than 0.1. This would be a unjustified extrapolation from a purely mathematical standpoint, but it seems reasonable in the light of the physical interpretation suggested in the preceding paragraph.

The effect of filters with other than flat frequency response has not been shown at all in terms of transient response. Two aspects of this question appear important. In the first place, it may be possible to obtain some improvement in transient response by appropriate filter design, but further analysis appears necessary to answer this question. In the second place, assuming that the flat filter gives a satisfactory response, the effect of high-frequency cutoff, which is inevitable in a practical system, must be estimated. A tentative answer to this question can be obtained by examination of the expressions for frequency response $P(s)$ developed for specific configurations. In all these expressions, the system response is substantially the same as in the flat-filter case as long as the filters $H_n(s)$ remain substantially flat until the frequency $s$ becomes large compared with the zero-frequency filter gains $\lambda_n$. This condition establishes an approximate bandwidth requirement for the filters. The extrapolation to arbitrary configurations is proposed in this case also.

The effect of departure from the reciprocity conditions is illustrated in only one case: the unilateral ring. Here, although the departure from reciprocity is the greatest possible, the effect on the transient response is mild. The magnitude of the response, aud its rate of subsidence, are substantially unchanged; the principal effect is tbe precession of the disturbance around the ring. The oscillatory components in the response can be associated with this precession.

Extrapolation of this result appears uncertain. The reciprocity condi-

tion can be stated in terms of the equality of the products of averaging coefficients in opposite directions around any loop. It can easily be conjectured that if the product of the averaging coefficients around any loop is much larger in one direction than the other, there will be a tendency for disturbances to precess around the loop in this direction, thereby generating oscillatory components in the response. On the other hand, it is hard to imagine pure precession in a multiloop network. A possible answer is suggested by the argument in Section III, in terms of pole loci, suggesting that a considerable departure from reciprocity could be tolerated before oscillatory components began to appear.

This extrapolation is suggested only for the case of flat or nearly flat filters and zero or small delays. For other cases, departures from reciprocity may give rise to a stability problem. This is suggested by the analysis in Section VIII of the discontinuities in the frequency response of an infinite unilateral ring, which showed that the stability condition that has been shown in the general case only to be a sufficient condition is in this case not merely sufficient but necessary. The latitude for filter shaping may be smaller in the nonreciprocal case, limited not simply by instability but by the deterioration of transient response that generally accompanies an approach to instability.

The analysis of jitter response shows that in certain representative cases the effect of jitter does not accumulate in a large system. This gives a definite negative answer to the question of whether cumulative jitter necessarily occurs in a large system. It seems reasonable to conjecture that this conclusion is independent of configuration, and remains true for substantially flat filters and small delays, but less reasonable to suppose that it will remain true for arbitrary filters.

Nothing in this study should be construed to indicate a preferred configuration for a practical system. Full or nearly full interconnections, nearest-neighbor connections, branching networks, or other forms may be appropriate. In particular, the apparent superiority of the fully interconnected network from the standpoint of transient response must be tempered by the practical considerations against setting up a large number of very long connections.

## XIII. ACKNOWLEDGMENTS

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## APPENDIX

### *Reciprocal Systems in the Steady State*

The assumption of zero initial conditions, used in the study of transient behavior, must now be dropped. Thus, the transformed (2) are no longer valid, but the original equations (1) may be used. In the steady state, the rate of change of phase at every station is equal to the common system frequency $f$,

$$p_n'(t) = f, \qquad n = 1, \cdots, N \tag{107}$$

so that

$$p_n(t) = ft + \psi_n, \qquad -\infty < t < \infty. \tag{108}$$

Thus, in the steady state, the $v_n(t)$ being constant, the system equations (1) become

$$f = v_n + \lambda_n \sum_{m=1}^{N} a_{nm}(\psi_m - \psi_n - f\tau_{nm}), \qquad n = 1, \cdots, N. \tag{109}$$

The general solution to these equations is the expression given by Gersho and Karafin[1] in terms of cofactors of a matrix derived from the $a_{nm}$. In the reciprocal case, let the $n$th equation in (109) be multiplied by $C_n$ and the equations be assumed over all $n$; when the reciprocity condition in the form of (8) is applied, all the terms in the phases $\psi_n$ drop out and one gets

$$f \sum_{n=1}^{N} C_n = \sum_{n=1}^{N} C_n v_n - f \sum_{n=1}^{N} \lambda_n C_n \sum_{m=1}^{N} a_{nm}\tau_{nm}. \tag{110}$$

This can be solved immediately for $f$, the expression being similar in form to the solution reported by Gersho and Karafin, except that the $C_n$, which are easily determined by (9), replace the matrix cofactors.

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